16 CIRCLES.
WHAT GOES AROUND...

This is the first of two lessons dealing with circles. This lesson gives some basic definitions and some elementary theorems, the most important of which is the Inscribed Angle Theorem. In the next lesson, we will tackle the important issue of circumference and see how that leads to the radian angle measurement system.

Definitions

So you might be thinking "Lesson 16 and we are just now getting to circles... what was the hold-up?" In fact, we could have given a proper definition for the term *circle* as far back as lesson 3. All that you really need for a good definition is points, segments, and congruence. But once you give the definition, what next? Most of what I want to cover with circles is specific to Euclidean geometry. I don't know that many theorems about circles in neutral geometry, and in the discussion thus far, the only time I remember that the lack of circles made things awkward was when we looked at cyclic polygons. In any case, now *is* the time, so

> DEF: CIRCLE
> For any point O and positive real number r, the *circle* with *center* O and *radius* r is the set of points which are a distance r from O.

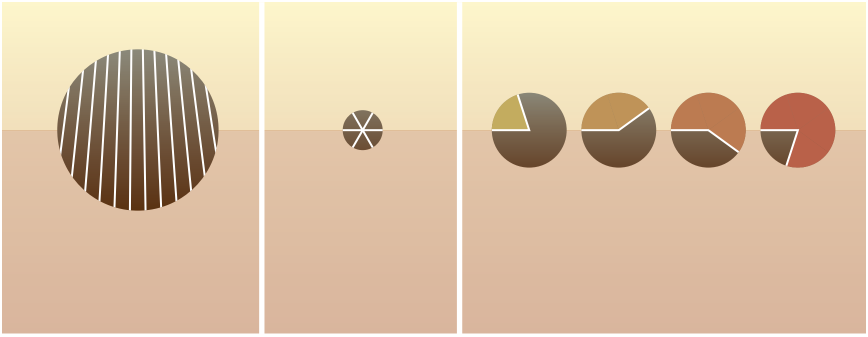
A few observations.

1. A circle is a set. Therefore, you should probably speak of the elements of that set as the points *of* the circle, but it is more common to refer to these as points *on* the circle.

2. In the definition I have given, the radius is a number. We often talk about the radius as a geometric entity though– as one of the segments from the center to a point on the circle.

3. We tend to think of the center of a circle as a fundamental part of it, but you should notice that the center of a circle is not actually a point on the circle.

4. It is not that common to talk about circles as congruent or not congruent. If you were to do it, though, you would say that two circles are congruent if and only if they have the same radius.



12 chords *3 diameters* *4 central angles*

Before we get into anything really complicated, let's get a few other related definitions out of the way.

DEF: CHORD AND DIAMETER

A segment with both endpoints on a circle is called a *chord* of that circle. A chord which passes through the center of the circle is called a *diameter* of that circle.

Just like the term radius, the term diameter plays two roles, a numerical one and geometric one. The diameter in the numerical sense is just the length of the diameter in the geometric sense.

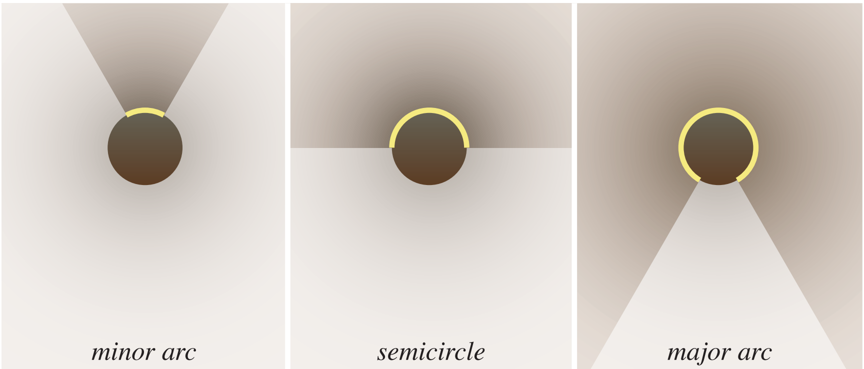
DEF: CENTRAL ANGLE

An angle with its vertex at the center of a circle is called a *central angle* of that circle.

We will see (in the next section) that a line intersects a circle at most twice. Therefore, if AB is a chord of a circle, then all the points of that circle other than A and B are on one side or the other of $\leftarrow AB \rightarrow$. Thus $\leftarrow AB \rightarrow$ separates those points into two sets. These sets are called *arcs* of the circle. There are three types of arcs– semicircles, major arcs, and minor arcs– depending upon where the chord crosses the circle.

DEF: SEMICIRCLE

Let AB be a diameter of a circle $\mathcal{C}$. All the points of $\mathcal{C}$ which are on one side of $\leftarrow AB \rightarrow$, together with the endpoints A and B, form a *semicircle*.



Each diameter divides the circle into two semicircles, overlapping at the endpoints A and B.

DEF: MAJOR AND MINOR ARC

Let AB be a chord of a circle $\mathcal{C}$ which is not a diameter, and let O be the center of this circle. All the points of $\mathcal{C}$ which are on the same side of $\leftarrow AB \rightarrow$ as O, together with the endpoints A and B, form a *major arc*. All the points of $\mathcal{C}$ which are on the opposite side of $\leftarrow AB \rightarrow$ from O, together with the endpoints A and B, form a *minor arc*.

Like the two semicircles defined by a diameter, the major and minor arcs defined by a chord overlap only at the endpoints A and B. For arcs in general, including diameters, I use the notation $\smile AB$. Most of the arcs we look at will be minor arcs, so in the instances when I want to emphasize that we are looking at a major arc, I will use the notation $\frown AB$.

There is a very simple, direct, and important relationship between arcs and central angles. You may recall that in the lesson on polygons, I suggested that two rays with a common endpoint define not one, but two angles– a "proper" angle and a "reflex" angle. These proper and reflex angles are related to the minor and major arcs as described in the next theorem, whose proof I leave to you.

THM: CENTRAL ANGLES AND ARCS

Let AB be a chord of a circle with center O. The points of $\smile AB$ are A, B, and all the points in the interior of the proper angle $\angle AOB$. The points of $\frown AB$ are A, B, and all the points in the interior of the reflex angle $\angle AOB$ (that is, the points exterior to the proper angle).

Intersections

Circles are different from the shapes we have been studying to this point because they are not built out of lines or line segments. Circles do share at least one characteristic with simple polygons though– they have an interior and an exterior. For any circle $\mathcal{C}$ with center O and radius r, and for any point P which is not $\mathcal{C}$,

- if $|OP| < r$, then P is inside $\mathcal{C}$;
- if $|OP| > r$, then P is outside $\mathcal{C}$.

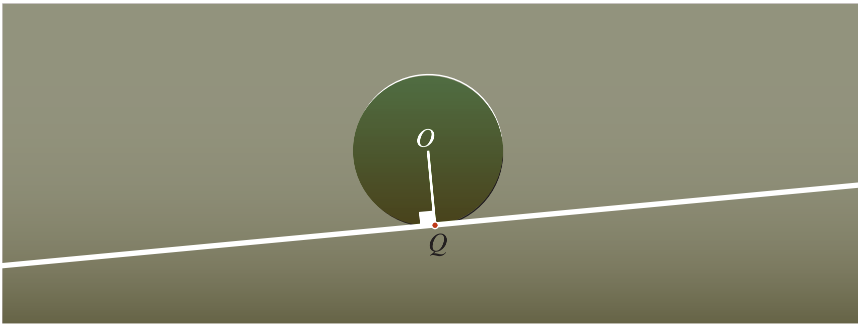
The set of points inside the circle is the interior and the set of points outside the circle is the exterior. Just like simple polygons, the circle separates the interior and exterior from each other. To get a better sense of that, we need to look at how circles intersect other basic geometric objects.

THM: A LINE AND A CIRCLE
A line will intersect a circle in 0, 1, or 2 points.

Proof. Let O be the center of a circle $\mathcal{C}$ of radius r, and let ℓ be a line. It is easy to find points on ℓ that are very far from $\mathcal{C}$, but are there any points on ℓ that are close to $\mathcal{C}$? The easiest way to figure out how close ℓ gets to $\mathcal{C}$ is to look at the closest point on ℓ to the center O. We saw (it was a lemma for the proof of A·A·A·S·S in lesson 10) that the closest point to O on ℓ is the foot of the perpendicular– call this point Q.

Zero intersections: $|OQ| > r$.
All the other points of ℓ are even farther from O, so none of the points on ℓ can be on $\mathcal{C}$.

One intersection: $|OQ| = r$.
Of course Q is an intersection, but it is the only intersection because all the other points on ℓ are farther away from O.



Two intersections: $|OQ| < r$.
The line spends time both inside and outside the circle. We just need to find where the line crosses in, and then back out of, the circle. The idea is to relate a point's distance from O to its distance from Q, and we can do that with the Pythagorean Theorem. If P is any point on ℓ other than Q, then $\triangle OQP$ will be a right triangle with side lengths that are related by the Pythagorean theorem

$$|OQ|^2 + |QP|^2 = |OP|^2.$$

In order for P to be on the circle, $|OP|$ must be exactly r. That means that $|PQ|$ must be exactly $\sqrt{r^2 - |OQ|^2}$. Since $|OQ| < r$, this expression is a positive real number, and so there are exactly two points on ℓ, one on each side of Q, that are this distance from Q. □

A line that intersects a circle once (at the foot of the perpendicular) is called a *tangent* line to the circle. A line that intersects a circle twice is called a *secant* line of the circle. There is a important corollary that turns this last theorem about lines into a related theorem about segments.

COR: A SEGMENT AND A CIRCLE
If point P is inside a circle, and point Q is outside it, then the segment PQ intersects the circle.

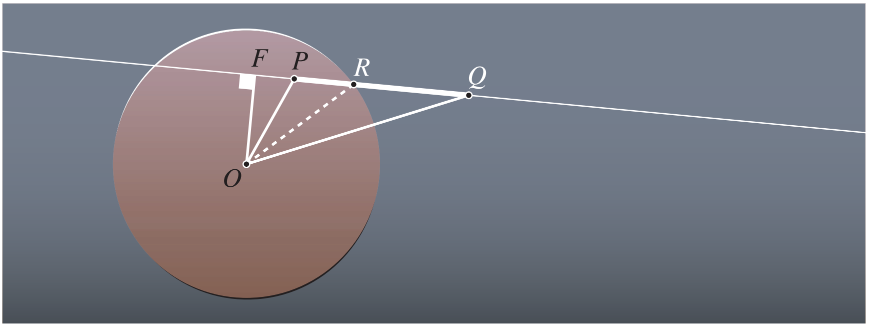
Proof. Label the center of the circle O. From the last theorem, we know that $\leftarrow PQ \rightarrow$ intersects the circle twice, and that the two intersections are separated by F, the foot of the perpendicular to PQ through O. The important intersection here is the one that is on the same side of the foot of the perpendicular as Q– call this point R. According to the Pythagorean theorem (with triangles $\triangle OFR$ and $\triangle OFQ$),

$$|FQ| = \sqrt{|OQ|^2 - |OF|^2} \quad \& \quad |FR| = \sqrt{|OR|^2 - |OF|^2}.$$

Since $|OQ| > |OR|$, $|FQ| > |FR|$, which places R between F and Q. We don't know whether P and Q are on the same side of F, though. If they are on opposite sides of F, then $P * F * R * Q$, so R is between P and Q as needed. If P and Q are on the same side of F, then we need to look at the right triangles $\triangle OFP$ and $\triangle OFR$. They tell us that

$$|FP| = \sqrt{|OP|^2 - |OF|^2} \quad \& \quad |FQ| = \sqrt{|OQ|^2 - |OF|^2}.$$

Since $|OP| < |OR|$, $|FP| < |FR|$, which places P between F and R. Finally, if P is between F and R, and R is between F and Q, then R has to be between P and Q. □



There is another important question of intersections, and that involves the intersection of two circles. If two circles intersect, then it is highly likely their two centers and the point of intersection will be the vertices of a triangle (there is a chance the three could be colinear, and we will deal with that separately). The lengths of all three sides of that triangle will be known (the two radii and the distance between centers). So this question is not so much one about circles, but whether triangles can be built with three given side lengths. We have one very relevant result– the Triangle Inequality says that if a, b, and c are the lengths of the side of a triangle, then

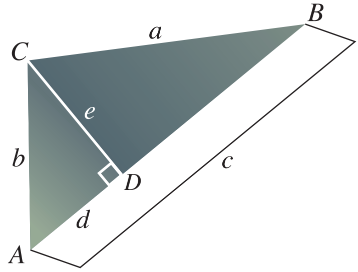
$$|a-b| < c < a+b.$$

What about the converse, though? If a, b, and c are any positive reals satisfying the Triangle Inequality conditions, can we put together a triangle with sides of those lengths? As much as a digression as it is, we need to answer this question before moving on.

THM: BUILDABLE TRIANGLES

Let a, b, and c be positive real numbers. Suppose that c is the largest of them and that $c < a+b$. Then there is a triangle with sides of length a, b, and c.

Proof. Start off with a segment AB whose length is c. We need to place a third point C so that it is a distance a from B and b from A. According to S·S·S, there is only one such triangle "up to congruence", so this may not be too easy. What I am going to do, though, is to build this triangle out of a couple of right triangles (so that I can use the Pythagorean theorem). Mark D on $AB\to$ and label $d = |AD|$. Mark C on one of the rays with endpoint D which is perpendicular to AB and label $e = |CD|$. Then both $\triangle ACD$ and $\triangle BCD$ are right triangles. Furthermore, by sliding D and C along their respective rays, we can make d and e any positive numbers.



We need to see if it is possible to position the two so that $|AC| = b$ and $|BC| = a$.

> To get $|AC| = b$, we will need $d^2 + e^2 = b^2$.
> To get $|BC| = a$, we will need $(c-d)^2 + e^2 = a^2$.

It's time for a little algebra to find d and e. According to the Pythagorean Theorem,

$$\begin{aligned} b^2 - d^2 = e^2 &= a^2 - (c-d)^2 \\ b^2 - d^2 &= a^2 - c^2 + 2cd - d^2 \\ b^2 &= a^2 - c^2 + 2cd \\ (b^2 - a^2 + c^2)/2c &= d. \end{aligned}$$

Since we initially required $c > a$, this will be a positive value. Now let's plug back in to find e.

$$e^2 = b^2 - d^2 = b^2 - \left(\frac{b^2 - a^2 + c^2}{2c}\right)^2.$$

Here is the essential part– because we will have to take a square root to find e, the right hand side of this equation has to be positive– otherwise the equation has no solution and the triangle cannot be built. Let's go back to see if the Triangle Inequality condition on the three sides will help:

$$\begin{gathered} c < a + b \\ c - b < a \\ (c-b)^2 < a^2 \\ c^2 - 2bc + b^2 < a^2 \\ c^2 - a^2 + b^2 < 2bc \\ (c^2 - a^2 + b^2)/2c < b \\ ((c^2 - a^2 + b^2)/2c)^2 < b^2 \\ 0 < b^2 - ((c^2 - a^2 + b^2)/2c)^2 \end{gathered}$$

which is exactly what we want [of course, when I first did this calculation, I worked in the other direction, from the answer to the condition]. As long as $c < a + b$, then, a value for e can be found, and that means the triangle can be built. □

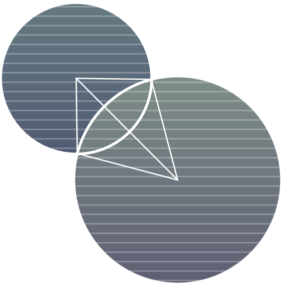
Now let's get back to the real issue at hand– that of the intersection of two circles.

THM: A CIRCLE AND A CIRCLE
Two circles intersect at 0, 1, or 2 points.

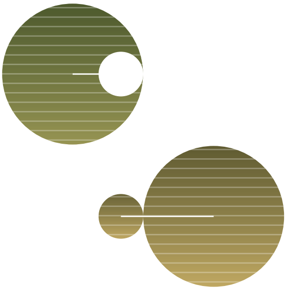
Proof. Three factors come in to play here: the radius of each circle and the distance between their centers. Label

r_1, r_2: the radii of the two circles, and
c, the distance between the centers.

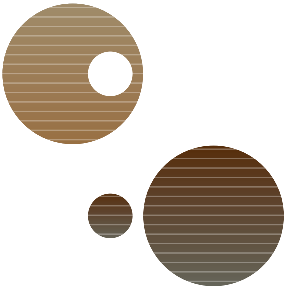
Two intersections:
when $|r_1 - r_2| < c < r_1 + r_2$.
There are exactly two triangles, $\triangle O_1XO_2$ and $\triangle O_1YO_2$, one on each side of O_1O_2, with sides of the required lengths. Therefore there are exactly two intersections of the two circles.



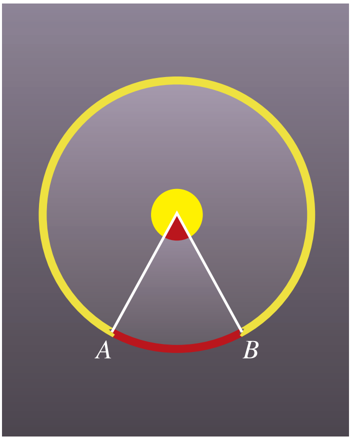
One intersection:
when $c = |r_1 - r_2|$ *or* $c = r_1 + r_2$.
In these two limiting cases, the triangle devolves into a line segment and the two intersections merge. In the first, either $O_1 * O_2 * X$ or $X * O_1 * O_2$, depending upon which radius is larger. In the second $O_1 * X * O_2$.



Zero intersections:
when $c < |r_1 - r_2|$ *or* $c > r_1 + r_2$.
In this case, you just cannot form the needed triangle (it would violate the Triangle Inequality), so there cannot be any intersections. In the first case, one circles lies entirely inside the other. In the second, they are separated from one another.



□



■ *Major arc: reflex* $\angle AOB$
■ *Minor arc: proper* $\angle AOB$

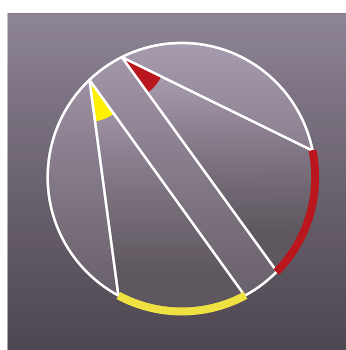
As I mentioned before, there is a one-to-one correspondence between central angles and arcs that matches the proper angle $\angle AOB$ with the minor arc $\smile AB$ and the reflex angle $\angle AOB$ with the major arc $\frown AB$. In the next lesson we are going to look at the relationship between the size of the central angle and the length of the corresponding arc (which is the basis for radian measure). In the meantime, I will use the correspondence as a way to simplify my illustrations– by using an arc to indicate a central angle, I can keep the picture from getting too crowded around the center of the circle.

The Inscribed Angle Theorem

In this section we will prove the Inscribed Angle Theorem, a result which is indispensible when working with circles. I suspect that this theorem is the most elementary result of Euclidean geometry which is generally *not* known to the average calculus student. Before stating the theorem, we must define an inscribed angle, the subject of the theorem.

DEF: INSCRIBED ANGLE
If A, B, and C are all points on a circle, then $\angle ABC$ is an *inscribed angle* on that circle.

Given any inscribed angle $\angle ABC$, points A and C are the endpoints of two arcs (either a minor and a major arc or two semicircles). Excluding the endpoints, one of those two arcs will be contained in the interior of $\angle ABC$ (a homework problem). We say, then, that $\angle ABC$ is inscribed on that arc. The Inscribed Angle Theorem describes the close relationship between an inscribed angle and the central angle on the same arc.



Two inscribed angles

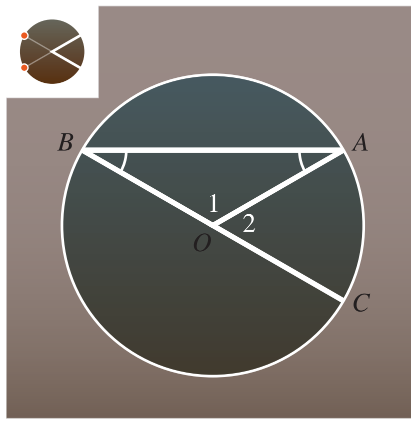
THE INSCRIBED ANGLE THEOREM

If $\angle BAC$ is an inscribed angle on a circle with center O, then

$$(\angle ABC) = \frac{1}{2}(\angle AOC).$$

Proof. This proof is a good lesson on the benefits of starting off with an easy case. There are three parts to this proof, depending upon the location of the vertex B relative to the lines OA and OC.

Part 1. When B is the intersection of $OC\rightarrow^{op}$ with the circle, or when B is the intersection of $OA \rightarrow^{op}$ with the circle.



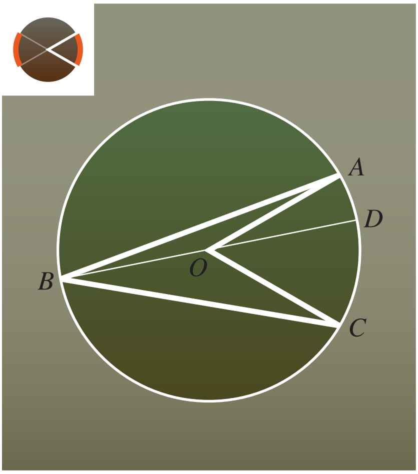
Even though we are only establishing the theorem for two very particular locations of B, this part is the key that unlocks everything else. Now, while I have given two possible locations for B, I am going to prove the result for just the first one (where B is on $OC\rightarrow^{op}$). All you have to do to prove the other part is to switch the letters A and C. Label $\angle AOB$ as $\angle 1$ and $\angle AOC$ as $\angle 2$. These angles are supplementary, so

$$(\angle 1) + (\angle 2) = 180^\circ. \quad (i)$$

The angle sum of $\triangle AOB$ is 180°, but in that triangle $\angle A$ and $\angle B$ are opposite congruent segments, so by the Isosceles Triangle Theorem they are congruent. Therefore

$$2(\angle B) + (\angle 1) = 180^\circ, \quad (ii)$$

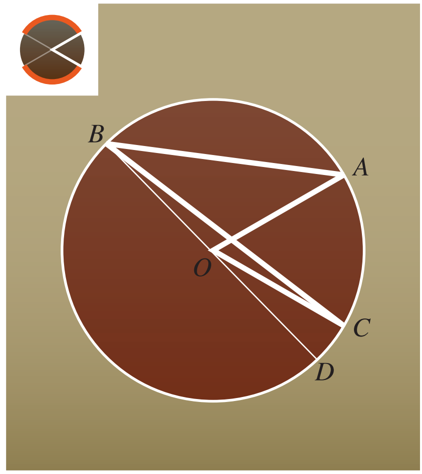
and if we subtract equation (ii) from equation (i), we get $(\angle 2) - 2(\angle B) = 0$, so $(\angle AOC) = 2(\angle ABC)$.



Part 2. When B is in the interior of $\angle AOC$, or when B is in the interior of the angle formed by $OA\rightarrow^{op}$ and $OC\rightarrow^{op}$, or when $A*O*C$.

There are three scenarios here– in the first the central angle is reflex, in the second it is proper, and in the third it is a straight angle– but the proof is the same for all of them. In each of these scenarios, the line $\leftarrow OB \rightarrow$ splits both the inscribed and the central angles. In order to identify these four angles, let me label one more point: D is the second intersection of $\leftarrow OB \rightarrow$ with the circle (so BD is a diameter of the circle). Using angle addition in conjunction with the previous results,

$$\begin{aligned}(\angle AOC) &= (\angle AOD) + (\angle DOC)\\ &= 2(\angle ABD) + 2(\angle DBC)\\ &= 2((\angle ABD) + (\angle DBC))\\ &= 2(\angle ABC).\end{aligned}$$



Part 3. When B is in the interior of the angle formed by $OA\rightarrow$ and $OC\rightarrow^{op}$, or when B is in the interior of the angle formed by $OC\rightarrow$ and $OA\rightarrow^{op}$.

As in the last case, label D so that BD is a diameter. The difference this time is that we need to use angle subtraction instead of angle addition. Since subtraction is a little less symmetric than addition, the two scenarios will differ slightly (in terms of lettering). In the first scenario

$$\begin{aligned}(\angle AOC) &= (\angle AOD) - (\angle DOC)\\ &= 2(\angle ABD) - 2(\angle DBC)\\ &= 2((\angle ABD) - (\angle DBC))\\ &= 2(\angle ABC).\end{aligned}$$

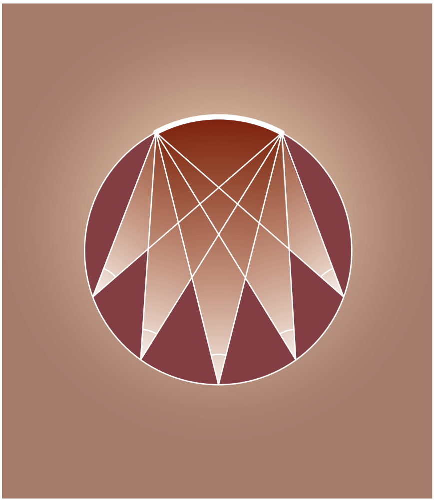
To get the second, you just need to switch A and C. $\square$

There are two important and immediate corollaries to this theorem. First, because all inscribed angles on a given arc share the same central angle,

> COR 1
> All inscribed angles on a given arc are congruent.

Second, the special case where the central angle $\angle AOC$ is a straight angle, so that the inscribed $\angle ABC$ is a right angle, is important enough to earn its own name

> THALES' THEOREM
> If C is a point on a circle with diameter AB (and C is neither A nor B), then $\triangle ABC$ is a right triangle.



Five congruent angles inscribed on the same arc.

A right angle inscribed on a semicircle.

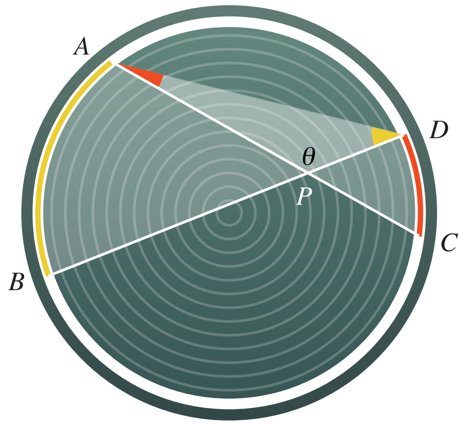
Applications of the Inscribed Angle Theorem

Using the Inscribed Angle Theorem, we can establish several nice relationships between chords, secants, and tangents associated with a circle. I will look at two of these results to end this lesson and put some more in the exercises.

THE CHORD-CHORD FORMULA

Let $\mathcal{C}$ be a circle with center O. Suppose that AC and BD are chords of this circle, and suppose further that they intersect at a point P. Label the angle of intersection, $\theta = \angle APD \simeq \angle BPC$. Then

$$(\theta) = \frac{(\angle AOD) + (\angle BOC)}{2}.$$



Proof. The angle θ is an interior angle of $\triangle APD$, so

$$(\theta) = 180^\circ - (\angle A) - (\angle D).$$

Both $\angle A$ and $\angle D$ are inscribed angles– $\angle A$ is inscribed on the arc $\smile CD$ and $\angle D$ is inscribed on the arc $\smile AB$. According to the Inscribed Angle Theorem, they are half the size of the corresponding central angles, so

$$\begin{aligned}(\theta) &= 180^\circ - \tfrac{1}{2}(\angle COD) - \tfrac{1}{2}(\angle AOB)\\ &= \tfrac{1}{2}(360^\circ - (\angle COD) - (\angle AOB)).\end{aligned}$$

This is some progress, for at least now θ is related to central angles, but alas, these are not the central angles in the formula. If we add all four central angles around O, though,

$$(\angle AOB) + (\angle BOC) + (\angle COD) + (\angle DOA) = 360^\circ$$
$$(\angle BOC) + (\angle DOA) = 360^\circ - (\angle COD) - (\angle AOB).$$

Now just substitute in, and you have the formula. □

According to the Chord-Chord formula, as long as the intersection point P is inside the circle, θ can be computed as the average of two central angles. What would happen if P moved outside the circle? Of course then we would not be talking about chords, since chords stop at the circle boundary, but rather the secant lines containing them.

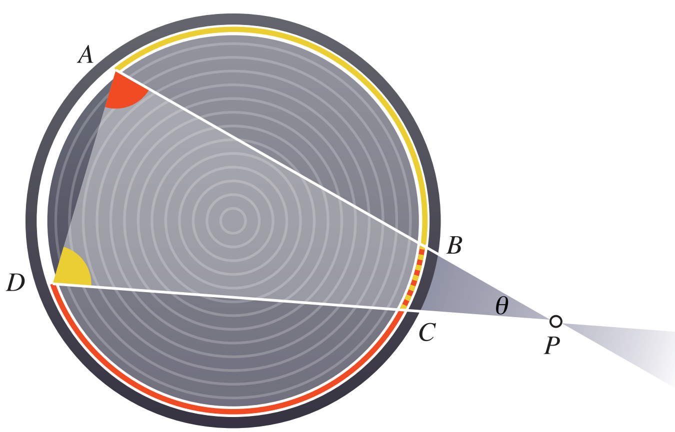
THE SECANT-SECANT FORMULA

Suppose that A, B, C, and D are points on a circle, arranged so that $\square ABCD$ is a simple quadrilateral, and that the secant lines AB and CD intersect at a point P which is outside the circle. Label the angle of intersection, $\angle APD$, as θ. If P occurs on the same side of AD as B and C, then

$$(\theta) = \frac{(\angle AOD) - (\angle BOC)}{2}.$$

If P occurs on the same side of BC as A and D, then

$$(\theta) = \frac{(\angle BOC) - (\angle AOD)}{2}.$$



Proof. There is obviously a great deal of symmetry between the two cases, so let me just address the first. The same principles apply here as in the last proof. Angle θ is an interior angle of $\triangle APD$, so

$$(\theta) = 180^\circ - (\angle A) - (\angle D).$$

Both $\angle A$ and $\angle D$ are inscribed angles– $\angle A$ is inscribed on arc $\smile BD$ and $\angle D$ is inscribed on arc $\smile AC$. We need to use the Inscribed Angle Theorem to relate these angles to central angles, and in this case, those central angles overlap a bit, so we will need to break them down further, but the rest is straightforward.

$$\begin{aligned}
(\theta) &= 180^\circ - \tfrac{1}{2}(\angle BOD) - \tfrac{1}{2}(\angle AOC)\\
&= \tfrac{1}{2}(360^\circ - (\angle BOD) - (\angle AOC))\\
&= \tfrac{1}{2}(360^\circ - (\angle BOC) - (\angle COD) - (\angle AOB) - (\angle BOC))\\
&= \tfrac{1}{2}([360^\circ - (\angle AOB) - (\angle BOC) - (\angle COD)] - (\angle BOC))\\
&= \tfrac{1}{2}((\angle AOD) - (\angle BOC)).
\end{aligned}$$

□

Exercises

1. Verify that the length of a diameter of a circle is twice the radius.
2. Prove that no line is entirely contained in any circle.
3. Prove that a circle is convex. That is, prove that if points P and Q are inside a circle, then all the points on the segment PQ are inside the circle.
4. Prove that for any circle there is a triangle entirely contained in it (all the points of the triangle are inside the circle).
5. Prove that for any circle there is a triangle which entirely contains it (all the points of the circle are in the interior of the triangle).
6. In the proof that two circles intersect at most twice, I have called both

   (1) $|a-b| < c < a+b$, and (2) $c \geq a, b$ and $c < a+b$

   the Triangle Inequality conditions. Verify that the two statements are equivalent for any three positive real numbers.
7. Let $\angle ABC$ be an inscribed angle on a circle. Prove that, excluding the endpoints, exactly one of the two arcs $\smile AC$ lies in the interior of $\angle ABC$.
8. Prove the converse of Thales' theorem: if $\triangle ABC$ is a right triangle with right angle at C, then C is on the circle with diameter AB.
9. Consider a simple quadrilateral which is inscribed on a circle (that is, all four vertices are on the circle). Prove that the opposite angles of this quadrilateral are supplementary.
10. Let C be a circle and P be a point outside of it. Prove that there are exactly two lines which pass through P and are tangent to C. Let Q and R be the points of tangency for the two lines. Prove that PQ and PR are congruent.
11. The "Tangent-Tangent" formula. Let P be a point which is outside of a circle $\mathcal{C}$. Consider the two tangent lines to $\mathcal{C}$ which pass through P and let A and B be the points of tangency between those lines and the circle. Prove that

$$(\angle APB) = \frac{(\angle 1) - (\angle 2)}{2}$$

where $\angle 1$ is the reflex central angle corresponding to the major arc $\frown AB$ and $\angle 2$ is the proper central angle corresponding to the minor arc $\smile AB$.

12. Let AC and BD be two chords of a circle which intersect at a point P inside that circle. Prove that

$$|AP| \cdot |CP| = |BP| \cdot |DP|.$$

References

I learned of the Chord-Chord, Secant-Secant, and Tangent-Tangent formulas in the Wallace and West book *Roads to Geometry*[1]. They use the names Two-Chord Angle Theorem, Two-Secant Angle Theorem, and Two-Tangent Angle Theorem.

[1] Edward C. Wallace and Stephen F. West. *Roads to Geometry*. Pearson Education, Inc., Upper Saddle River, New Jersey, 3rd edition, 2004.